Chaudhary Mahadeo Prasad Degree College

(A CONSTITUENT PG COLLEGE OF UNIVERSITY OF ALLAHABD)

Subject: Physics

(Study material for Post Graduate students)

M. Sc. III Semester

Paper: Fourth

Course Code: PHY 658

Quantum States of Radiation

Unit I: Coherent States of Radiation and their Properties, Coherent State as wave packet, Expansion of States and Operators in Terms of Coherent States

Prepared by

Dr. Rakesh Kumar

DEPARTMENT OF PHYSICS

Recommended Books:

- Quantum Optics: M. O. Scully and M. S. Zubairy
- Optical Coherence and Quantum Optics: L. Mandel and E. Wolf
- Introduction to Modern Quantum Optics: J S Peng and G X Li
- Quantum Optics: D F Walls and G J Milburn
- Principle of Optics: M Born and E Wolf
- Quantum Optics: M. Fox
- Introductory Quantum Optics: C C Gerry
- Quantum Optics: G S Agarwal
- Mathematical Methods of Quantum Optics: R R Puri
- Concepts of Quantum Optics: P L Knight and L Allen

PAPER – IV: QUANTUM STATES OF RADIATION (SYLLABUS)

PHY658

Unit I Coherent States of Radiation and their Properties, Coherent State as wave packet, Expansion of States and Operators in Terms of Coherent States.

Unit II Density Operator of Radiation, Sudarshan-Glauber Representation, Density Operators of Coherent and Chaotic Radiation, Coherence and Characteristics Functions.

Unit III Polarization and Stokes Parameters, Annihilation and Creation Operators for Modes with General Polarization, Unpolarized Light.

Unit IV Photoelectron Counting Distribution, Hanbury Brown and Twiss Experiment, Bunching and Antibunching, Example of pure Fock State for Antibunching of Photons,

Unit V Schwartz Inequalities and Quantum Behaviour of Optical Fields, Squeezed States of Radiation (Elementary Discussion)

Introduction of annihilation, creation operator and Occupation number states

I Quantum condition for system of particle:

$$\left.\begin{aligned}&[q_m, q_n] = 0\,;\ [p_m, p_n] = 0\,;\\ &[q_m, p_n] = i\delta_{mn}\end{aligned}\right\} \quad (1)$$

II Quantum condition for system field:

$$\left.\begin{aligned}&[\psi(\vec{r},t), \psi(\vec{r}',t)] = 0\,;\ [\psi^\dagger(\vec{r},t), \psi^\dagger(\vec{r}',t)] = 0\\ &[\psi(\vec{r},t), \psi^\dagger(\vec{r}',t)] = \delta^3(\vec{r}-\vec{r}')\end{aligned}\right\} \quad (2)$$

Non Relativistic Schrödinger wave equation

Hamiltonian Formalism

$$H = \int d^3\vec{r}\,[\frac{1}{2m}\nabla\psi^*\nabla\psi + V\psi^*\psi] \quad (3)$$

Here, ψ is wave function

$$H = \int d^3\vec{r}\,[\frac{1}{2m}\overline{\nabla}\psi^\dagger\overline{\nabla}\psi + V\psi^\dagger\psi] \quad (4)$$

Here, ψ is field operator and $\psi^\dagger$ is Hermitian conjugate of ψ.

Non Relativistic Schrodinger equation is

$$i\dot{\psi} = [\psi, H] \quad (5)$$

$$\boxed{i\dot{\psi} = -\frac{1}{2m}\nabla^2\psi + V\psi}$$

Expansion of ψ, $\psi^\dagger$ in terms of eigen function u_k and other complex conjugates:-

Consider u_k (k =1, 2, 3, ………) ⟶ complete family of orthonormal eigen function of system of particles or field.

$$\left.\begin{aligned}&\text{Orthonormality condition:} \quad \int d^3\vec{r}\ u_k^* u_l = \delta_{kl}\\ &\text{Completeness relation:} \quad \sum_k u_k(r) u_k^*(\vec{r}') = \delta^3(\vec{r}-\vec{r}')\end{aligned}\right\} \quad (6)$$

Expansion of ψ: $\psi(\vec{r},t) = \sum_k a_k(t) u_{k(\vec{r})}$ (7)

Where u_k is c-number and $a_k(t)$ is q-number.

Eq.(7) can be solved to obtained $a_k(t)$

$$\int d^3\vec{r}\ \psi(\vec{r},t)\ u_k^*(\vec{r}) = \int d^3\vec{r} \sum_l a_l(t)u_{l(\vec{r})}\ u_k^*(\vec{r}) \qquad (8)$$

$$\int d^3\vec{r}\ \psi(\vec{r},t)\ u_k^*(\vec{r}) = \sum_l a_l(t)\delta_{kl} = a_k(t) \qquad (9)$$

Hermitian Conjugate of Eq.(9) is

$$a_k^\dagger(t) = \int d^3\vec{r}'\ \psi^\dagger(\vec{r},t)\ u_{k'}(\vec{r}') \qquad (10)$$

$$\begin{aligned}[a_k(t), a_k^\dagger(t)] &= \int d^3\vec{r}\ d^3\vec{r}' u_k(\vec{r})u_k(\vec{r}')[\psi(\vec{r},t), \psi^\dagger(\vec{r}',t)] \\ &= \int d^3\vec{r}\ d^3\vec{r}' u_k(\vec{r})u_{k'}(\vec{r}')\ \delta^3(\vec{r}-\vec{r}') \\ &= \int d^3\vec{r}\ d^3\vec{r}' u_k(\vec{r})u_{k'}(\vec{r}) \\ &= \delta_{kk'}\end{aligned} \qquad (11)$$

Hence,

$$\left.\begin{aligned}[a_k(t), a_{k'}(t)] &= 0 \\ [a_k^\dagger(t), a_{k'}^\dagger(t)] &= 0 \\ [a_k(t), a_{k'}^\dagger(t)] &= \delta_{kk'}\end{aligned}\right\} \qquad (12)$$

Number Operator:

$\psi(\vec{r},t) = \sum_k a_k(t)u_{k(\vec{r})}$ and $\psi^\dagger(\vec{r},t) = \sum_k a_k^\dagger(t)u_{k(\vec{r})}^*$

From Eq.(4)

$$H = \int d^3\vec{r}[\frac{1}{2m}\vec{\nabla}\psi^\dagger\vec{\nabla}\psi + V\psi^\dagger\psi]$$

If u_k's are the eigen function with eigen values E_k then

$$H = \sum_k E_k a_k^\dagger a_k$$

H is the operator for number of particle in mode k,

$$\boxed{H = \sum_k E_k N_k}$$

Where $N_k = a_k^\dagger a_k$ and the number operator $N = \sum_k N_k$

For single mode, $[a, a^\dagger] = 1$ and $N = a^\dagger a$

Occupation number state:

If $|n\rangle$ is the eigen state of number operator then,

$N|n\rangle = n|n\rangle$;

Orthonormality condition: $\langle n|m\rangle = \delta_{nm}$;

Completeness relation: $\sum_n |n\rangle\langle n| = 1$.

These states are known as occupation number states.

Commutation relation for a, N and interpretation of a:

$$[a,\ N] = [a,\ a^\dagger a] = [a,\ a^\dagger]a = a$$

$$aN - Na = a$$

$$Na = a(N-1)$$

In general $f(N)a = a\, f(N-1)$

Interpretation of a: If $|n\rangle$ is the occupation number state with n particles then

$$N|n\rangle = n|n\rangle$$

$$\begin{aligned} N\big(a|n\rangle\big) &= a(N-1)|n\rangle \\ &= a(n-1)|n\rangle \\ &= (n\text{-}1)\big(a|n\rangle\big) \end{aligned}$$

‘a’ decreases the occupation number by 1. So, it is called ***annihilation operator***.

Commutation relation for a, N and interpretation of a†:

$$[a^\dagger,\ N] = [a^\dagger,\ a^\dagger a] = a^\dagger[a^\dagger,\ a] = -a^\dagger$$

$$a^\dagger N - Na^\dagger = -a^\dagger$$

$$Na^\dagger = a^\dagger(N+1)$$

In general $f(N)a^\dagger = a^\dagger\, f(N+1)$

Interpretation of a: If $|n\rangle$ is the occupation number state with n particles

$$N|n\rangle = n|n\rangle$$

$$\begin{aligned} N\big(a^\dagger|n\rangle\big) &= a^\dagger(N+1)|n\rangle \\ &= a^\dagger(n+1)|n\rangle \\ &= (n+1)\big(a^\dagger|n\rangle\big) \end{aligned}$$

‘a†’ increases the occupation number by 1. So, it is called ***creation operator***.

Action of $|n\rangle$ on a and $a^\dagger$:

Since ‘a’ is annihilation operator so we can write

$$a|n\rangle = \alpha|n-1\rangle \qquad (13)$$

Hermitian conjugate of Eq.(13) is

$$\langle n|a^{\dagger} = \langle n-1|\alpha^{*} \qquad (14)$$

Scalar Product of (13) and (14) is

$$\langle n|a^{\dagger} a|n\rangle = \langle n-1|\alpha^{*}\alpha|n-1\rangle \;;\; N = a^{\dagger}a$$

$$\langle n|N|n\rangle = |\alpha|^{2}\langle n-1|n-1\rangle \;;\; N|n\rangle = n|n\rangle$$

$$n\langle n|n\rangle = |\alpha|^{2}\langle n-1|n-1\rangle$$

$$n = |\alpha|^{2} ; \langle n|n\rangle = \langle n-1|n-1\rangle = 1$$

$$\alpha = \sqrt{n}$$

Hence

$$\boxed{a|n\rangle = \sqrt{n}|n-1\rangle}$$

Similary if ' $a^{\dagger}$ ' is creation operator then we can write

$$a^{\dagger}|n\rangle = \beta|n+1\rangle \qquad (15)$$

Hermitian conjugate of Eq.(15) is

$$\langle n|a = \langle n+1|\beta^{*} \qquad (16)$$

Scalar Product of Eq.(15) and Eq.(16) is

$$\langle n|aa^{\dagger}|n\rangle = \langle n+1|\beta^{*}\beta|n+1\rangle \;;\; N+1 = aa^{\dagger}$$

$$\langle n|N+1|n\rangle = |\beta|^{2}\langle n+1\|n+1\rangle \;;\; N|n\rangle = n|n\rangle$$

$$(n+1)\langle n\|n\rangle = |\beta|^{2}\langle n+1\|n+1\rangle$$

$$n+1 = |\beta|^{2} ; \langle n\|n\rangle = \langle n+1\|n+1\rangle = 1$$

$$\beta = \sqrt{n+1}$$

Hence

$$\boxed{a^{\dagger}|n\rangle = \sqrt{n+1}|n+1\rangle}$$

Generalization to multimode:

$$[a_{k}, a_{k'}] = 0$$

$$[a_{k}^{\dagger}, a_{k'}^{\dagger}] = 0$$

$$[a_k, a_{k'}^{\dagger}] = \delta_{kk'}$$

.

$$|\{n_k\}\rangle = |n_1, n_2, n_3, n_4............., n_k, ...\rangle$$

$$N_k|\{n_k\}\rangle = n_k|n_1, n_2, n_3, n_4............., n_k, ...\rangle$$

$$a_k|\{n_k\}\rangle = \sqrt{n_k}|n_1, n_2, n_3, n_4............., n_k - 1, ...\rangle$$

$$a_k^{\dagger}|\{n_k\}\rangle = \sqrt{n_k + 1}|n_1, n_2, n_3, n_4............., n_k + 1, ...\rangle$$

COHERENT STATE ($|\alpha\rangle$):

It is an eigenstate of annihilation operator 'a'.

$$a|\alpha\rangle = \alpha|\alpha\rangle \tag{17}$$

α be complex number

$$\alpha = \alpha_r + i\alpha_i = |\alpha|e^{i\theta_\alpha} \tag{18}$$

Relation between $|\alpha\rangle$ and $|n\rangle$:

We know that $|\alpha\rangle = \sum_{n=0}^{\infty} c_n|n\rangle$,

using equation (17)

$$a \sum_{n=0}^{\infty} c_n|n\rangle = \alpha \sum_{n=0}^{\infty} c_n|n\rangle \; ; \; \sum_{n=0}^{\infty} c_n a|n\rangle = \sum_{n=0}^{\infty} c_n \alpha|n\rangle$$

$$\because a|n\rangle = \sqrt{n}|n-1\rangle$$

$$\sum_{n=1}^{\infty} c_n \sqrt{n}|n-1\rangle = \alpha \sum_{n=0}^{\infty} c_n|n\rangle$$

$$\sqrt{1}\, c_1|0\rangle + \sqrt{2}\, c_2|1\rangle + \sqrt{3}\, c_3|2\rangle + ... = \alpha\, c_0|0\rangle + \alpha\, c_1|1\rangle + \alpha\, c_2|2\rangle + ...$$

Comparing the coefficients of the occupation number states on both sides,

$$|0\rangle \Rightarrow c_1 = \frac{\alpha\, c_0}{\sqrt{1}}$$

$$|1\rangle \Rightarrow c_2 = \frac{\alpha\, c_1}{\sqrt{2}} = \frac{\alpha \cdot \alpha\, c_0}{\sqrt{2}.\sqrt{1}} = \frac{\alpha^2\, c_0}{\sqrt{1.2}}$$

$$|2\rangle \Rightarrow c_3 = \frac{\alpha\, c_2}{\sqrt{3}} = \frac{\alpha\, \alpha^2\, c_0}{\sqrt{3}.\sqrt{2}.\sqrt{1}} = \frac{\alpha^3\, c_0}{\sqrt{1.2.3}}$$

and so on

$$|n\rangle \Rightarrow c_n = \frac{\alpha\, c_2}{\sqrt{3}} = \frac{\alpha\,\alpha^2\, c_0}{\sqrt{3}.\sqrt{2}.\sqrt{1}} = \frac{\alpha^3\, c_0}{\sqrt{1.2.3}} = ... = \frac{\alpha^n\, c_0}{\sqrt{1.2.3.4..........n}} = \frac{\alpha^n\, c_0}{\sqrt{n!}}$$

Hence

$$|\alpha\rangle = \sum_{n=0}^{\infty} \frac{\alpha^n\, c_0}{\sqrt{n!}} |n\rangle$$

$$|\alpha\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Evaluation of c_0 **:**

Use normalization condition,

$$\langle\alpha|\alpha\rangle = 1$$

$$1 = |c_0|^2 \sum_{n=0}^{\infty} \frac{|\alpha|^{2n}}{n!} = |c_0|^2 e^{|\alpha|^2}$$

$$|c_0|^2 = e^{-|\alpha|^2}\ ;\ c_0 = e^{-\frac{|\alpha|^2}{2}}$$

Hence

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

Note:

If $|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}} |0\rangle$

Then $|\alpha\rangle = \exp(-\frac{|\alpha|^2}{2}) \sum_{n=0}^{\infty} \frac{(\alpha a^\dagger)^n}{n!} |0\rangle$

$$|\alpha\rangle = \exp(-\frac{|\alpha|^2}{2}) \exp(\alpha a^\dagger) |0\rangle$$

$\exp(-\frac{|\alpha|^2}{2})\exp(\alpha a^\dagger)$ is not hermitian. The coherent state $|\alpha\rangle$ can also be generated by displacing the vacuum $|0\rangle$

$$|\alpha\rangle = D(\alpha)|0\rangle$$

where $D(\alpha)$ is called **displacement operator** defined by ,

$$D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$$

is hermitian. The relation $|\alpha\rangle = D(\alpha)|0\rangle$ is important as it show how the coherent state can be generated..

Important Properties of the displacement operator

(1) $\therefore |\alpha\rangle = D(\alpha)|0\rangle$

Proof: If A and B are operator and satisfies the following :

$$[[A,B],A] = [[A,B],B] = 0$$

then $$e^{A+B} = e^{A}e^{B}e^{-\frac{1}{2}[A,B]} = e^{A}e^{B} = e^{A+B+\frac{1}{2}[A,B]}$$

is called Baker Campbell Housdroff Identity (BCH).

From above $\exp(\alpha a^{\dagger} - \alpha^{*}a)$, consider A= $\alpha a^{\dagger}$ and B = - $\alpha^{*}a$, then using BCH identity

$$\exp(\alpha a^{\dagger} - \alpha^{*}a) = e^{\alpha a^{\dagger}}e^{-\alpha^{*}a}e^{-\frac{1}{2}[\alpha a^{\dagger}, -\alpha^{*}a]}$$

$$= e^{\alpha a^{\dagger}}e^{-\alpha^{*}a}e^{\frac{1}{2}\alpha\alpha^{*}[a^{\dagger},a]}$$

Since $[a^{\dagger},a] = -1$

Therefore $\exp(\alpha a^{\dagger} - \alpha^{*}a) = e^{\alpha a^{\dagger}}e^{-\alpha^{*}a}e^{-\frac{1}{2}|\alpha|^2} = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha a^{\dagger}}e^{-\alpha^{*}a}$

i.e $D(\alpha) = \exp(\alpha a^{\dagger} - \alpha^{*}a) = e^{\alpha a^{\dagger}}e^{-\alpha^{*}a}e^{-\frac{1}{2}|\alpha|^2} = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha a^{\dagger}}e^{-\alpha^{*}a}$

$$D(\alpha)|0\rangle = \exp(\alpha a^{\dagger} - \alpha^{*}a)|0\rangle = e^{\alpha a^{\dagger}}e^{-\alpha^{*}a}e^{-\frac{1}{2}|\alpha|^2}|0\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha a^{\dagger}}e^{-\alpha^{*}a}|0\rangle$$

$$e^{-\alpha^{*}a}|0\rangle = (1 - \frac{\alpha^{*}a}{1!} + \frac{\alpha^{*2}a^{2}}{2!} - ....)|0\rangle$$

$$= |0\rangle - \frac{\alpha^{*}a}{1!}|0\rangle + \frac{\alpha^{*2}a^{2}}{2!}|0\rangle - ....$$

$$= |0\rangle \qquad \text{since } (a|0\rangle = 0)$$

so $$D(\alpha)|0\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha a^{\dagger}}e^{-\alpha^{*}a}|0\rangle = e^{-\frac{1}{2}|\alpha|^2}e^{\alpha a^{\dagger}}|0\rangle \qquad (1)$$

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle$$

$$\therefore |\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle = e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}\frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle = e^{-\frac{|\alpha|^2}{2}}e^{\alpha a^{\dagger}}|0\rangle \qquad (2)$$

From (1) and (2)

$$\therefore |\alpha\rangle = D(\alpha)|0\rangle$$

Hence $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-\frac{1}{2}|\alpha|^2} = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a}$ is hermitian.

(2) $\exp(\alpha a^\dagger - \alpha^* a) = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{\frac{1}{2}|\alpha|^2} e^{-\alpha^* a} e^{\alpha a^\dagger}$

Proof: We know that the displacement operator is $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$,

Consider A= $\alpha a^\dagger$ and B = - $\alpha^* a$, then using BCH identity

$$\exp(\alpha a^\dagger - \alpha^* a) = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-\frac{1}{2}[\alpha a^\dagger, -\alpha^* a]}$$

$$= e^{\alpha a^\dagger} e^{-\alpha^* a} e^{\frac{1}{2}\alpha\alpha^*[a^\dagger, a]}$$

Since $[a^\dagger, a] = -1$

Therefore $\exp(\alpha a^\dagger - \alpha^* a) = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-\frac{1}{2}|\alpha|^2} = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a}$

i.e $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) = e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-\frac{1}{2}|\alpha|^2} = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a}$

Now, if we consider A= - $\alpha^* a$ and B = $\alpha a^\dagger$ and use then use BCH identity we get,

$$\exp(-\alpha^* a + \alpha a^\dagger) = e^{-\alpha^* a} e^{\alpha a^\dagger} e^{-\frac{1}{2}[-\alpha^* a, \alpha a^\dagger]}$$

$$= e^{\alpha a^\dagger} e^{-\alpha^* a} e^{\frac{1}{2}\alpha\alpha^*[a, a^\dagger]}$$

Since $[a, a^\dagger] = 1$

Therefore $\exp(-\alpha^* a + \alpha a^\dagger) = e^{-\alpha^* a} e^{\alpha a^\dagger} e^{\frac{1}{2}|\alpha|^2} = e^{\frac{1}{2}|\alpha|^2} e^{-\alpha^* a} e^{\alpha a^\dagger}$

hence $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a) = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a} = e^{\frac{1}{2}|\alpha|^2} e^{-\alpha^* a} e^{\alpha a^\dagger}$

(3) $D(\alpha)D(\beta) = \exp[\tfrac{1}{2}(\alpha\beta^* - \alpha^*\beta)]D(\alpha+\beta)$

Proof: Since $D(\alpha) = \exp(\alpha a^\dagger - \alpha^* a)$ and $D(\beta) = \exp(\beta a^\dagger - \beta^* a)$

$$D(\alpha)D(\beta) = \exp(\alpha a^\dagger - \alpha^* a)\exp(\beta a^\dagger - \beta^* a)$$

$$D(\alpha)D(\beta) = \exp\{(\alpha+\beta)a^\dagger - (\alpha^* + \beta^*)a\}$$

Using BCH identity $\qquad e^{A+B} = e^{A}e^{B}e^{-\frac{1}{2}[A,B]}$

$$A = (\alpha+\beta)a^{\dagger}\,,\; B = -(\alpha^{*}+\beta^{*})a$$

$$D(\alpha)D(\beta) = \exp\left\{(\alpha+\beta)a^{\dagger} - (\alpha^{*}+\beta^{*})a\right\}$$

$$= e^{(\alpha+\beta)a^{\dagger}} e^{-(\alpha^{*}+\beta^{*})a} e^{-\frac{1}{2}[(\alpha+\beta)a^{\dagger},-(\alpha^{*}+\beta^{*})a]}$$

$$= e^{(\alpha+\beta)a^{\dagger}} e^{-(\alpha^{*}+\beta^{*})a} e^{\frac{1}{2}[(\alpha+\beta)a^{\dagger},(\alpha^{*}+\beta^{*})a]}$$

Now calculate $e^{\frac{1}{2}[(\alpha+\beta)a^{\dagger},(\alpha^{*}+\beta^{*})a]}$,

$$e^{\frac{1}{2}[(\alpha+\beta)a^{\dagger},(\alpha^{*}+\beta^{*})a]} = \exp\{\tfrac{1}{2}([\alpha a^{\dagger},\alpha^{*}a]\backslash+[\alpha a^{\dagger},\beta^{*}a]+[\beta a^{\dagger},\alpha^{*}a]+[\beta a^{\dagger},\beta^{*}a]\}$$

$$= \exp\{\tfrac{1}{2}(|\alpha|^{2}[a^{\dagger},a]\backslash+\alpha\beta^{*}[a^{\dagger},a]+\beta\alpha^{*}[a^{\dagger},a]+|\beta|^{2}[a^{\dagger},a]\}$$

$$= e^{-\frac{1}{2}|\alpha|^{2}-\frac{1}{2}|\beta|^{2}-\frac{1}{2}\alpha\beta^{*}-\frac{1}{2}\beta\alpha^{*}}$$

$$= e^{-\frac{1}{2}|\alpha+\beta|^{2}}\, e^{\frac{1}{2}(\alpha\beta^{*}-\beta\alpha^{*})}$$

$$D(\alpha)D(\beta) = \exp\left\{(\alpha+\beta)a^{\dagger} - (\alpha^{*}+\beta^{*})a\right\} = e^{(\alpha+\beta)a^{\dagger}} e^{-(\alpha^{*}+\beta^{*})a} e^{\frac{1}{2}[(\alpha+\beta)a^{\dagger},(\alpha^{*}+\beta^{*})a]}$$

$$= e^{(\alpha+\beta)a^{\dagger}} e^{-(\alpha^{*}+\beta^{*})a} e^{-\frac{1}{2}|\alpha+\beta|^{2}} e^{\frac{1}{2}(\alpha\beta^{*}-\beta\alpha^{*})}$$

$$= \exp\{(\alpha+\beta)a^{\dagger} - (\alpha^{*}+\beta^{*})a\} e^{-\frac{1}{2}|\alpha+\beta|^{2}} e^{\frac{1}{2}(\alpha\beta^{*}-\beta\alpha^{*})}$$

$$= e^{\frac{1}{2}(\alpha\beta^{*}-\beta\alpha^{*})}\, D(\alpha+\beta)$$

Solve the following:

(i) $D^{\dagger}(\xi)\, a\, D(\xi) = a + \xi$

(ii) $D^{\dagger}(\xi)\, a^{\dagger}\, D(\xi) = a^{\dagger} + \xi^{*}$

(iii) $D^{-1}(\xi) = D(\xi) = D(-\xi)$

Properties of Coherent States:

(1) Coherent states are not orthogonal.

Proof:

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n\rangle$$

$$|\beta\rangle = e^{-\frac{|\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{\beta^n}{\sqrt{n!}} |n\rangle$$

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2 - 2\alpha^*\beta\right)}$$

(i) If $\alpha = \beta$ then $\langle\alpha|\beta\rangle = 1$ coherent states are normalized.

(ii) If $\alpha \neq \beta$ then

$$\langle\alpha|\beta\rangle = e^{-\frac{1}{2}\left(|\alpha|^2 + |\beta|^2 + 2\alpha^*\beta\right)}$$

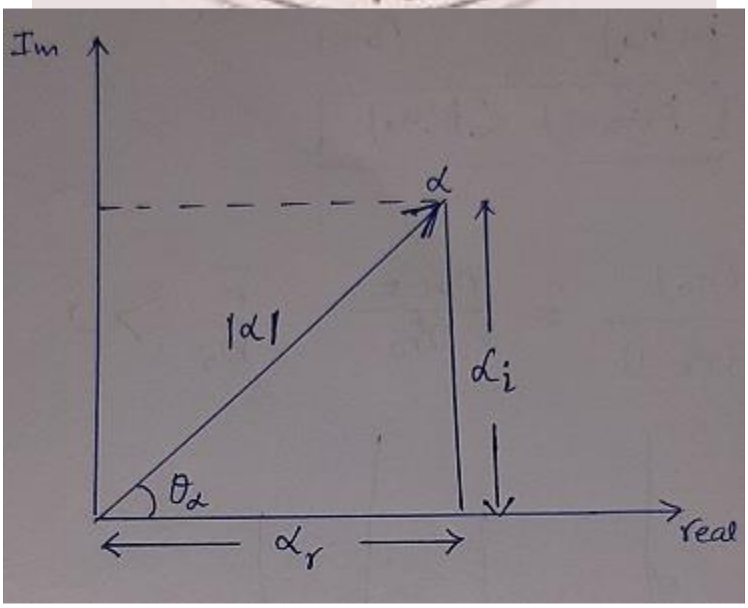
$$= e^{-\frac{1}{2}\left(|\alpha - \beta|^2\right)} \neq 0$$

Hence Coherent states are not orthogonal, it may be orthogonal for $|\alpha - \beta| >> 1$. For large separation of the eigen value i.e. $|\alpha - \beta| >> 1$, $|\alpha - \beta|^2 \to 0$ then $\langle\alpha|\beta\rangle = 0$.

(2) Coherent states are overcomplete.

Proof: Consider $\int d^2\alpha |\alpha\rangle\langle\alpha|$

$d^2\alpha \to$ represents the *surface element in complex plane like ds in real plane.*



$$ds = dxdy = Rdrd\theta$$

$$d^2\alpha = d\alpha_r d\alpha_i = |\alpha| d|\alpha| d\theta_\alpha$$

$$\int d^2\alpha |\alpha\rangle\langle\alpha| = \int d^2\alpha\, e^{-|\alpha|^2} \sum_{n,m} \frac{\alpha^n \alpha^{*m}}{\sqrt{n!}\sqrt{m!}} |n\rangle\langle m|$$

$$= \int_0^\infty |\alpha| d|\alpha| \int_0^{2\pi} d\theta_\alpha\, e^{-|\alpha|^2} \sum_{n,m} \frac{|\alpha|^{n+m} e^{i(n-m)\theta_\alpha}}{\sqrt{n!}\sqrt{m!}} |n\rangle\langle m|$$

$\because \quad \alpha = |\alpha|\, e^{i\theta_\alpha}$

$$\int_0^{2\pi} e^{i(n-m)\theta_\alpha} d\theta_\alpha = 2\pi\delta_{nm}$$
$$= 2\pi \quad \text{for } n = m$$
$$= 0 \text{ for } n \neq m$$

$$\int d^2\alpha |\alpha\rangle\langle\alpha| = 2\pi \int_0^\infty |\alpha| d|\alpha| e^{-|\alpha|^2} \sum_n \frac{|\alpha|^{2n}}{n!} |n\rangle\langle n|$$

If we write $|\alpha| = X$ then

$$\int d^2\alpha |\alpha\rangle\langle\alpha| = 2\pi \int_0^\infty X dX e^{-X^2} \sum_n \frac{X^{2n}}{n!} |n\rangle\langle n|$$
$$= 2\pi \int_0^\infty dX e^{-X^2} \sum_n \frac{X^{2n+1}}{n!} |n\rangle\langle n|$$

$\because \qquad \int_0^\infty X^{2n+1} e^{-|\alpha|^2} dX = \frac{n!}{2}$

$$\int d^2\alpha |\alpha\rangle\langle\alpha| = \pi \sum_n |n\rangle\langle n| = \pi$$

Hence

$$\boxed{\frac{1}{\pi}\int d^2\alpha |\alpha\rangle\langle\alpha| = 1}$$

This is completeness relation for coherent state.

Over completeness of Coherent states: Overcompleteness of coherent states means that any state can be written in terms of the coherent state.

(i) We can write $|\alpha\rangle$ and $\langle\alpha|$ in an infinite number of ways.

(ii) We can resolve any state $|\alpha\rangle$ in terms of coherent states.

$$|\alpha\rangle = |\alpha\rangle.1 = |\alpha\rangle \frac{1}{\pi}\int d^2\beta |\beta\rangle\langle\beta|$$
$$= \frac{1}{\pi}\int d^2\beta \langle\beta|\alpha\rangle |\beta\rangle$$
$$= \frac{1}{\pi}\int d^2\beta \exp[-\tfrac{1}{2}(|\alpha|^2 + |\beta|^2 - 2\beta^*\alpha)] |\beta\rangle$$

(3) Expansion of states and operators in terms of the coherent states

Completeness relation for coherent state is $\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = 1$

This gives,

$$|\psi\rangle = \frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha|\psi\rangle$$
$$= \frac{1}{\pi}\int d^2\alpha\langle\alpha|\psi\rangle|\alpha\rangle$$

similary for operator, F

$$F = \frac{1}{\pi}\int d^2\alpha\ |\alpha\rangle\langle\alpha|\, F\, \frac{1}{\pi}\int d^2\beta\ |\beta\rangle\langle\beta|$$
$$= \frac{1}{\pi^2}\int d^2\alpha d^2\beta\langle\alpha|F|\beta\rangle|\alpha\rangle\langle\beta|$$

If $$|\psi\rangle = \sum_{n=0}^{\infty} C_n|n\rangle$$

then

$$|\psi\rangle = C_0|0\rangle + C_1|1\rangle + C_2|2\rangle + C_3|3\rangle + ------$$
$$= \left(C_0 + C_1\frac{a^{\dagger}}{\sqrt{1!}} +_1 \frac{a^{\dagger 2}}{\sqrt{2!}} +_1 ------\right)|0\rangle$$

$$\text{i.e. } |\psi\rangle = f(a^{\dagger})|0\rangle$$

$$a^{\dagger n}|0\rangle = a^{\dagger n-1}a^{\dagger}|0\rangle = a^{\dagger n-1}\sqrt{1}|1\rangle$$
$$= a^{\dagger n-2}\sqrt{1}\sqrt{2}|2\rangle$$
$$= a^{\dagger n-3}\sqrt{1}\sqrt{2}\sqrt{3}|3\rangle$$
$$= ---- \text{and so on}$$
$$= \sqrt{1.2.3.4......n}|n\rangle$$
$$= \sqrt{n!}|n\rangle$$

$$|n\rangle = \frac{a^{\dagger n}}{\sqrt{n!}}|0\rangle$$

Evaluate: $$\langle\alpha|\psi\rangle = \langle\alpha|\left(C_0 + C_1\frac{a^{\dagger}}{\sqrt{1!}} +_1 \frac{a^{\dagger 2}}{\sqrt{2!}} + ------\right)|0\rangle$$

$$\langle\alpha|\psi\rangle = \left(C_0 + C_1\frac{\alpha^*}{\sqrt{1!}} +_1 \frac{\alpha^{*2}}{\sqrt{2!}} + ------\right)\langle\alpha|0\rangle$$

Since $\langle\alpha|0\rangle = e^{-\frac{1}{2}|\alpha|^2}$

$$\langle\alpha|\psi\rangle = f(\alpha^*)\, e^{-\frac{1}{2}|\alpha|^2}$$

$$|\psi\rangle = \frac{1}{\pi}\int d^2\alpha |\alpha\rangle\langle.\alpha|\psi\rangle$$

$$= \frac{1}{\pi}\int d^2\alpha\ f(\alpha^*)\, e^{-\frac{1}{2}|\alpha|^2} |\alpha\rangle$$

This is the expansion of $|\psi\rangle$ in terms of coherent state.

(4) Coherent states are minimum uncertainty states:

Analogy of single mode radiation with a harmonic oscillator, in radiation gauge, $\vec{\nabla}\cdot\vec{A} = 0$, where $\vec{A}$ is vector potential then $\vec{A}$ can be written as

$$\vec{A} = \sum_{k,\lambda} \frac{1}{\sqrt{2\omega_k V}} \varepsilon_{k,\lambda}\left[a_{k,\lambda} e^{i(\vec{k}.\vec{r}-\omega t)} + a^{+}_{k,\lambda} e^{-i(\vec{k}.\vec{r}-\omega t)} \right]$$

In natural unit $\hbar = c = 1$,

K is wave vector,

ε is unit vector

λ denotes polarization (takes 2 values corresponds to 2 transverse mode of polarization)

V denotes Interaction Volume

$$\vec{E} = -\vec{\nabla}\phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}$$

$$B = \vec{\nabla}\times\vec{A}$$

$$\vec{E} = -\vec{\nabla}A_0 - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\ ;\ A^{\mu} = (\phi, \vec{A})$$

For single mode, vector potential

$$\vec{A} = \frac{1}{\sqrt{2\omega V}}\varepsilon\left[a_k e^{i(\vec{k}.\vec{r}-\omega t)} + a^{+}_k e^{-i(\vec{k}.\vec{r}-\omega t)} \right]$$

Hamiltonian: $H = \int d^3\vec{r}\,\frac{1}{8\pi}[\dot{\vec{A}}^2 + (\vec{\nabla}\times\vec{A})^2]$

This reduces to $H = \frac{\omega}{2}(a^{\dagger}a + aa^{\dagger}) = \omega(a^{\dagger}a + \frac{1}{2})$. This Hamiltonian is for single mode radiation.

Define two hermitian operators be $(a^{\dagger} + a)$ and $i(a^{\dagger} - a)$.

$$q = \frac{1}{\sqrt{2\omega}}(a^{\dagger} + a) \text{ - coordinate operator}$$

$$p = i\sqrt{\frac{\omega}{2}}(a^{\dagger} - a) \text{ - momentum operator}$$

which shows $[q,\ p] = i$.

$$q^2 = \frac{1}{2\omega}(a^{\dagger} + a)^2 = \frac{1}{2\omega}(a^{\dagger 2} + a^{\dagger}a + aa^{\dagger} + a^2)$$

$$p^2 = -\frac{\omega}{2}(a^{\dagger} - a)^2 = -\frac{\omega}{2}(a^{\dagger 2} - a^{\dagger}a - aa^{\dagger} + a^2)$$

$$\omega^2 q^2 + p^2 = \omega(a^{\dagger}a + aa^{\dagger})$$

$$H = \frac{1}{2}(\omega^2 q^2 + p^2) = \frac{\omega}{2}(a^{\dagger}a + aa^{\dagger}) = \omega(a^{\dagger}a + \tfrac{1}{2})$$

So for one dimensional harmonics oscillator, $[q,p] = i$, $H = \frac{1}{2m}(\omega^2 q^2 + p^2)$

Hence single mode radiation is equivalent to unit mass one dimensional harmonic oscillator. Thus annihilation and creation operator can be written in terms of coordinate and position operator.

$$a = \frac{1}{\sqrt{2\omega}}(\omega q + ip)$$

$$a^{\dagger} = \frac{1}{\sqrt{2\omega}}(\omega q - ip)$$

Uncertainties in q and p for coherent state:

Let Δq and Δp be the uncertainty in q (position) and p (momentum) respectively. Δq and Δp are square root of the variances of q and p are defined as follows:

$$\Delta q = \sqrt{\langle q^2 \rangle - \langle q \rangle^2}$$

$$\Delta p = \sqrt{\langle p^2 \rangle - \langle p \rangle^2}$$

For $q = \frac{1}{\sqrt{2\omega}}(a^{\dagger} + a)$

$$\langle q\rangle = \langle\alpha|q|\alpha\rangle$$
$$= \langle\alpha|\frac{1}{\sqrt{2\omega}}(a^\dagger + a)|\alpha\rangle$$
$$= \frac{1}{\sqrt{2\omega}}\left\{\langle\alpha|a^\dagger|\alpha\rangle + \langle\alpha|a|\alpha\rangle\right\}$$
$$= \frac{1}{\sqrt{2\omega}}\left\{\alpha^* + \alpha\right\} \qquad \text{Since } a|\alpha\rangle = \alpha|\alpha\rangle \text{ and } \langle\alpha|a^\dagger = \langle\alpha|\alpha^*$$

Since $\alpha = \alpha_r + i\alpha_i$ then $\alpha^* + \alpha = 2\,\alpha_r$ hence

$$\langle q\rangle = \frac{2\,\alpha_r}{\sqrt{2\omega}} = \sqrt{\frac{2}{\omega}}\,\alpha_r$$

$$\langle q^2\rangle = \langle\alpha|q^2|\alpha\rangle$$
$$= \langle\alpha|\frac{1}{2\omega}(a^\dagger + a)^2|\alpha\rangle$$
$$= \frac{1}{2\omega}\left\{\langle\alpha|a^{\dagger 2}|\alpha\rangle + \langle\alpha|a^2|\alpha\rangle + \langle\alpha|(aa^\dagger + a^\dagger a)|\alpha\rangle\right\}$$
$$= \frac{1}{2\omega}\left\{\langle\alpha|a^{\dagger 2}|\alpha\rangle + \langle\alpha|a^2|\alpha\rangle + \langle\alpha|(2a^\dagger a + 1)|\alpha\rangle\right\}$$
$$= \frac{1}{2\omega}\left\{\alpha^{*2} + \alpha^2 + 2\alpha^*\alpha + 1\right\}$$
$$= \frac{1}{2\omega}\left\{(\alpha^* + \alpha)^2 + 1\right\}$$
$$= \frac{1}{2\omega}\left\{4\,\alpha_r^2 + 1\right\} = \frac{2\,\alpha_r^2}{\omega} + \frac{1}{2\omega}$$

hence variance of q

$$(\Delta q)^2 = \langle q^2\rangle - \langle q\rangle^2 = \frac{1}{2\omega}$$

Similarly we can calculate

$$\langle p\rangle = \sqrt{2\omega}\;\alpha_i\;,\;\; \langle p^2\rangle = 2\,\omega\alpha_i^2 + \frac{\omega}{2}\text{ .and}$$

variance of p ; $(\Delta p)^2 = \langle p^2\rangle - \langle p\rangle^2 = \frac{\omega}{2}$.Thus,

$$(\Delta q)^2(\Delta p)^2 = \frac{1}{4} \text{ So, } (\Delta q)(\Delta p) = \frac{1}{2}.$$

This is the minimum value allowed by Heisenberg uncertainty principle. Hence, Coherent state is a minimum uncertainty state. It is the most classical state.

(5) Poisson distribution of Photons:

Let operator Ω has several eigen values ω_n corresponding to eigen states $|\omega_n\rangle$

$$\Omega|\omega_n\rangle = \omega_n|\omega_n\rangle$$

If we measure the physical values of operator Ω, we get the variables ω_n, different values corresponds to n = 0, 1, 2, 3, ……..

Suppose we have a mixed state,

$$|\psi\rangle = c_0|\omega_0\rangle + c_1|\omega_1\rangle + c_2|\omega_2\rangle + c_3|\omega_3\rangle + ...$$

Then $|\psi\rangle$ is an arbitrary or mixed state can be expanded linearly in terms of a complete set of orthonormal states ω_n. Probability that the measurement of ω_0 is $|c_0|^2$, ω_1 is $|c_1|^2$ and so on probability that the measurement of ω_n is $|c_n|^2$.

Coherent state is $|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle$ so it can be written as

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}}|0\rangle + e^{-\frac{|\alpha|^2}{2}}\frac{\alpha}{\sqrt{1!}}|1\rangle + e^{-\frac{|\alpha|^2}{2}}\frac{\alpha^2}{\sqrt{2!}}|2\rangle + ...$$

$$P(0) = e^{-|\alpha|^2},\ P(1) = e^{-|\alpha|^2}\frac{\alpha^2}{1!},\ P(2) = e^{-|\alpha|^2}\frac{\alpha^4}{2!},.........P(n) = e^{-|\alpha|^2}\frac{\alpha^{2n}}{n!}$$

Now,

$$\begin{aligned}
|\alpha\rangle &= 1.|\alpha\rangle \\
&= \sum_{n0}^{\infty}|n\rangle\langle n||\alpha\rangle \\
&= (|0\rangle\langle 0| + |1\rangle\langle 1| + |2\rangle\langle 2| + |3\rangle\langle 3| + ... + |n\rangle\langle n|)|\alpha\rangle. \\
&= \langle 0|\alpha\rangle|0\rangle + \langle 1|\alpha\rangle|1\rangle + \langle 2|\alpha\rangle|2\rangle + ... + \langle n|\alpha\rangle|n\rangle \\
&= c_0|0\rangle + c_1|1\rangle + c_2|2\rangle + ... + c_n|n\rangle
\end{aligned}$$

$$P(0) = |c_0|^2 = |\langle 0|\alpha\rangle|^2 = e^{-|\alpha|^2}$$

$$P(1) = |c_1|^2 = |\langle 1|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^2}{1!}$$

$$P(2) = |c_2|^2 = |\langle 2|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^4}{2!}$$

…………………………….

$$P(n) = |c_n|^2 = |\langle n|\alpha\rangle|^2 = e^{-|\alpha|^2}\frac{|\alpha|^{2n}}{n!}$$

Hence $$P(n) = e^{-|\alpha|^2}\frac{\alpha 2n}{n!}$$

$$\langle\alpha|a^+a|\alpha\rangle = \alpha^*\alpha = |\alpha|^2 = \bar{n} \text{ = average number of photons.}$$

Thus the expectation value of number of photons is the average number of photons in the coherent state.

$$P(n) = e^{-\bar{n}}\frac{(\bar{n})^n}{n!},$$

which is ***Poisson distribution*** for number of photons in the coherent state

$$\langle\alpha|a^{+2}a^2|\alpha\rangle = \alpha^{*2}\alpha^2 = |\alpha|^4 = \bar{n}^2$$

The variance of photon in coherent state is

$$\langle(\Delta n)^2\rangle = \langle\alpha|(a^+a)^2|\alpha\rangle - (\langle\alpha|a^+a|\alpha\rangle)^2$$
$$= \langle\alpha|a^+aa^+a|\alpha\rangle - (\langle\alpha|a^+a|\alpha\rangle)^2$$
$$= \langle\alpha|a^+(1+a^+a)a|\alpha\rangle - (\langle\alpha|a^+a|\alpha\rangle)^2$$
$$= \langle\alpha|a^{+2}a^2|\alpha\rangle + (\langle\alpha|a^+a|\alpha\rangle) - (\langle\alpha|a^+a|\alpha\rangle)^2$$
$$= |\alpha|^4 + |\alpha|^2 - |\alpha|^4$$
$$= |\alpha|^2 = \bar{n}$$

Thus for *Poisson distribution both variance and mean are equal.*

For large values of n Poisson distribution approaches a Gaussian distribution.

Note:

$$P(n) = e^{-\bar{n}}\frac{(\bar{n})^n}{n!},\ P(n+1) = e^{-\bar{n}}\frac{(\bar{n})^{n+1}}{(n+1)!}$$

Therefore $$\frac{P(n+1)}{P(n)} = \frac{\bar{n}}{n+1}$$

Case (1) If $\bar{n}$ is an integer

(i) For n= $\bar{n}$

$$\frac{P(\bar{n}+1)}{P(\bar{n})} = \frac{\bar{n}}{\bar{n}+1}$$

$$(\bar{n}+1)\,P(\bar{n}+1) = \bar{n}\,P(\bar{n})$$

$$P(\bar{n}+1) = \frac{\bar{n}\,P(\bar{n})}{(\bar{n}+1)}$$

Since $\frac{\bar{n}}{(\bar{n}+1)} < 1$

Therefore $P(\bar{n}+1) < P(\bar{n})$ If $\bar{n}$ is an integer

(ii) For n= $\bar{n}$ - 1

$$\frac{P(\bar{n})}{P(\bar{n}-1)} = \frac{e^{-\bar{n}}\bar{n}^{n}}{\bar{n}!}\frac{(\bar{n}-1)}{e^{-\bar{n}}\bar{n}^{n-1}} = 1$$

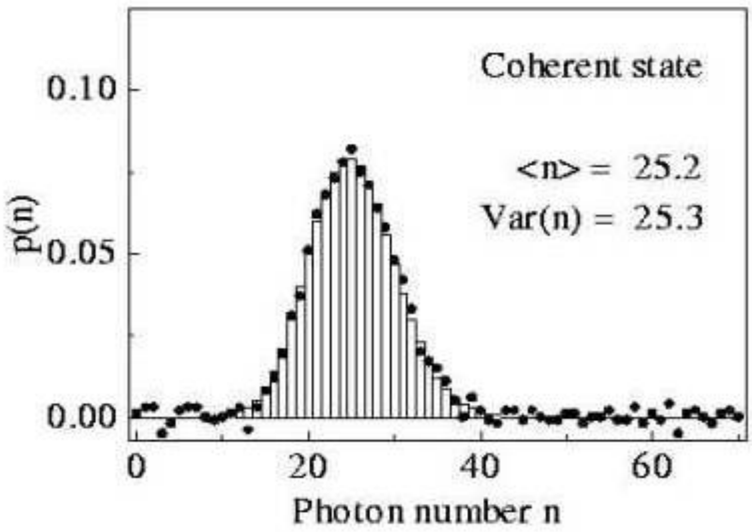
$$P(\bar{n}+1) = P(\bar{n})$$

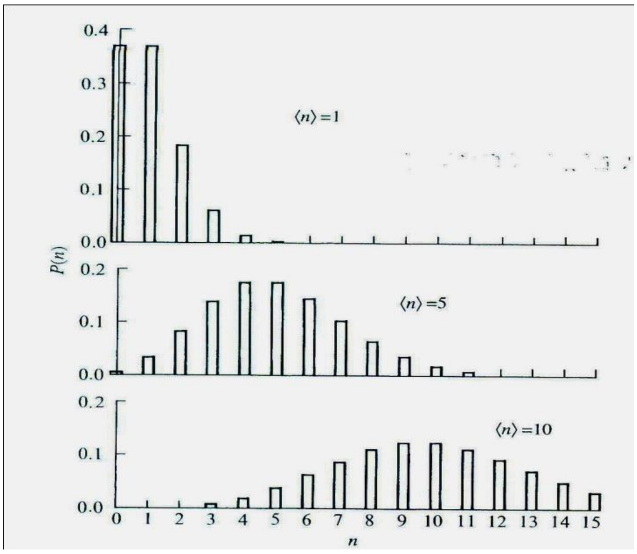
(iii) For n= $\bar{n}$ - 2

$$\frac{P(\bar{n}-1)}{P(\bar{n}-2)} = \frac{\bar{n}}{\bar{n}-1} > 1$$

$$P(\bar{n}-1) > P(\bar{n}-2)$$

Conclusion: If $\bar{n}$ is an integer out of P(0), P(1), ..etc. $P(\bar{n})$. $P(\bar{n})$ is the highest.





Case (2) If $\bar{n}$ is not an integer

For n = $\bar{n}+\varepsilon$; ε is a fraction <1.

(i) For n = $\bar{n}_0$

$$\frac{P(n_0+1)}{P(n_0)}=\frac{n_0+\varepsilon}{n_0+1}<1.$$

$$P(n_0+1)<P(n_0)$$

(ii) For n = $\bar{n}_0$

$$\frac{P(n_0)}{P(n_0-1)}=\frac{n_0+\varepsilon}{n_0}=\frac{\bar{n}}{n_0}>0.$$

$$P(n_0-1)<P(n_0)$$

(6) Show that $a^{\dagger}|\alpha\rangle=\frac{\partial}{\partial\alpha}|\alpha\rangle+\alpha^{*}|\alpha\rangle$

Proof:

$$|\alpha\rangle=e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$e^{\frac{|\alpha|^2}{2}}|\alpha\rangle=\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$$

$$\frac{\partial}{\partial\alpha}(e^{\frac{|\alpha|^2}{2}}|\alpha\rangle)=\frac{\partial}{\partial\alpha}(\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle)$$

$$=\sum_{n=0}^{\infty}\frac{n\alpha^{n-1}}{\sqrt{n!}}|n\rangle$$

$$=\sum_{n=1}^{\infty}\frac{\alpha^{n-1}}{\sqrt{(n-1)!}}\sqrt{n}|n\rangle$$

Replace n by n + 1

$$\frac{\partial}{\partial\alpha}\left(e^{\frac{|\alpha|^2}{2}}|\alpha\rangle\right)=\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{(n)!}}\sqrt{n+1}|n+1\rangle$$

$$=\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{(n)!}}a^+|n\rangle$$

$$=a^+\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{(n)!}}|n\rangle$$

$$=a^+\left(e^{\frac{|\alpha|^2}{2}}|\alpha\rangle\right)$$

$$=e^{\frac{|\alpha|^2}{2}}a^+|\alpha\rangle$$

$$a^+|\alpha\rangle=e^{-\frac{|\alpha|^2}{2}}\frac{\partial}{\partial\alpha}\left(e^{\frac{|\alpha|^2}{2}}|\alpha\rangle\right)$$

$$=\frac{\partial}{\partial\alpha}(|\alpha\rangle)+\alpha^*|\alpha\rangle$$

(7) Show that $e^{i\theta a^\dagger a}|\alpha\rangle=|\alpha e^{i\theta}\rangle$

Proof:

$$e^{i\theta a^\dagger a}|\alpha\rangle=e^{i\theta N}|\alpha\rangle$$

Since $|\alpha\rangle=e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle$

$$e^{i\theta N}|\alpha\rangle=e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{i\theta N}|n\rangle$$

Hence,

$$e^{i\theta N}|\alpha\rangle=e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}e^{i\theta n}|n\rangle\qquad\because e^{i\theta N}|n\rangle=e^{i\theta n}|n\rangle$$

$$=e^{-\frac{|\alpha|^2}{2}}\sum_{n=0}^{\infty}\frac{(\alpha e^{i\theta})^n}{\sqrt{n!}}|n\rangle$$

$$=|\alpha e^{i\theta}\rangle$$

If $\theta=\frac{\pi}{2}$ then

$$e^{i\theta N}|\alpha\rangle=e^{i\frac{\pi}{2}N}|\alpha\rangle=\left|\alpha e^{i\frac{\pi}{2}}\right\rangle=|i\alpha\rangle$$

Prove the following:

(1) $\langle\alpha|2\alpha\rangle=e^{-\frac{1}{2}|\alpha|^2}$

(2) $(\langle\alpha| + \langle-\alpha|)a(|\alpha\rangle + |-\alpha\rangle) = 0$

(3) $\langle\alpha|[a^2, a^{+2}]|\alpha\rangle = 4|\alpha|^2 + 2$

(4) $[X_1, X_2] = \frac{i}{2}$, where hermitian operators $X_{1,2}$ are defined by $a = X_1 + i\,X_2$

(5) $(\langle 2\alpha| + \langle\alpha|)(|-2\alpha\rangle - |-\alpha\rangle) = 0$

(6) $\int d^2\alpha |\alpha\rangle\langle-\alpha| = \pi\sum_{n0}^{\infty}(-1)^n |n\rangle\langle n|$

Coherent State as a Gaussian Wave Packet

Let $|X\rangle$ is the eigenstate when the position of the particle in precisely stated by X then

$$X_{op}|X\rangle = \chi|X\rangle$$

Where X_{op} is position operator

$$\psi(x) = \langle X|\psi\rangle$$

(i) Orthonormality condition: $\langle X'|X\rangle = \delta(X - X')$

(ii) Completeness relation: $\int dX|X\rangle\langle X| = 1$

(iii) Mean position- given by the expectation value

$$\begin{aligned}\langle\psi|X_{op}|\psi\rangle &= \langle\psi|X_{op}\int dX|X\rangle\langle X||\psi\rangle \\ &= \int dX\langle\psi|X_{op}|X\rangle\langle X|\psi\rangle \\ &= \int dX\langle\psi|\chi|X\rangle\langle X|\psi\rangle \\ &= \int dX\ \chi\langle\psi|X\rangle\langle X|\psi\rangle \\ &= \int dX\ \chi\psi^*(X)\psi(X)\end{aligned}$$

which gives position probability density $\psi^*(X)\psi(X)$. The Configuration space function for the coherent state $|\alpha\rangle$, defined by $\langle q'|\alpha\rangle$,

If $|q'\rangle$ is the eigen state, then $q|q'\rangle = q'|q'\rangle$

q = coordinate operator, it is quantum number and q' = eigen value, it is c-number

We know that $a|0\rangle = 0$. For single mode Harmonic Oscillator $a = \frac{1}{\sqrt{2\omega}}(\omega q + ip)$.

$$\langle q'|\frac{1}{\sqrt{2\omega}}(\omega q + ip)|0\rangle = 0$$

$$\langle q'|(\omega q + ip)|0\rangle = 0$$

$$\omega q'\langle q'|0\rangle + i\,p\langle q'|0\rangle = 0$$

Since $p = -i\frac{\partial}{\partial q'}$

$$\omega q'\langle q'|0\rangle + i\,(-i\frac{\partial}{\partial q'})\langle q'|0\rangle = 0$$

$$\omega q'\langle q'|0\rangle + \frac{\partial}{\partial q'}\langle q'|0\rangle = 0$$

$$\frac{\partial}{\partial q'}\langle q'|0\rangle = -\,\omega q'\langle q'|0\rangle,$$

which is similar to the differential equation $\frac{\partial y}{\partial x} = -\,a\,x\,y$ on solving we get,

$$y = Ae^{-\frac{1}{2}ax^2}$$

So $\langle q'|0\rangle = Ae^{-\frac{1}{2}\omega q'^2}$; A is constant of integration.

Using normalization condition

$$\int_{-\infty}^{\infty} dq' \left|\langle q'|0\rangle\right|^2 = 1$$

$$|A|^2 \int_{-\infty}^{\infty} dq'\, e^{-\omega q'^2} = 1$$

$|A|^2\sqrt{\frac{\pi}{\omega}} = 1$ which implies $A = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}}$, hence

$$\langle q'|0\rangle = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\omega q'^2}$$

Coherent state is

$$|\alpha\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}}|n\rangle\ ,$$

it can be written as $|\alpha\rangle = \exp(-\frac{|\alpha|^2}{2})\exp(\alpha a^\dagger)|0\rangle$

since

$$e^{\alpha a}|0\rangle = (1+\frac{\alpha a}{1!}+\frac{\alpha^2 a^2}{2!}-.........)\ |0\rangle$$

$$=|0\rangle+\frac{\alpha a}{1!}|0\rangle+\frac{\alpha^2 a^2}{2!}|0\rangle-........$$

$$=|0\rangle \qquad \text{since } (a|0\rangle = 0)$$

Hence $|\alpha\rangle = \exp(-\frac{|\alpha|^2}{2})\exp(\alpha a^\dagger)|0\rangle = \exp(-\frac{|\alpha|^2}{2})\exp(\alpha a^\dagger)\exp(\alpha a)|0\rangle$

Use BCH Identity

$$e^A e^B = e^{A+B} e^{\frac{1}{2}[A,B]}$$

$$A = \exp(\alpha a^\dagger) \text{ and } B = \exp(\alpha a)$$

$$e^{\alpha a^\dagger} e^{\alpha a} = e^{\alpha(a^\dagger + a)} e^{-\frac{1}{2}\alpha^2}$$

Hence

$$|\alpha\rangle = e^{-\frac{1}{2}|\alpha|^2} e^{\alpha(a^\dagger + a)} e^{-\frac{1}{2}\alpha^2} |0\rangle$$

$$= e^{-\frac{1}{2}(|\alpha|^2+\alpha^2)} e^{\alpha(a^\dagger + a)} |0\rangle$$

$$= e^{-\frac{1}{2}(|\alpha|^2+\alpha^2)} e^{\alpha\sqrt{2\omega}q'} |0\rangle$$

$$\langle q'|\alpha\rangle = e^{-\frac{1}{2}(|\alpha|^2+\alpha^2)} e^{\alpha\sqrt{2\omega}q'} \langle q'|0\rangle$$

$$\langle q'|0\rangle = \left(\frac{\pi}{\omega}\right)^{\frac{1}{4}} e^{-\frac{1}{2}\omega q'^2}$$

therefore

$$\langle q'|\alpha\rangle = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}(|\alpha|^2+\alpha^2)} e^{\alpha\sqrt{2\omega}q'} e^{-\frac{1}{2}\omega q'^2}$$

$$\langle q'|\alpha\rangle = \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} e^{-\frac{1}{2}(|\alpha|^2+\alpha^2)} e^{\alpha\sqrt{2\omega}q'} e^{-\frac{1}{2}\omega q'^2}$$

$$= \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} \exp\left[-\tfrac{1}{2}\left(|\alpha_r|^2 + |\alpha_i|^2 + \alpha_r^2 + \alpha_i^2 + \alpha_r^2 - \alpha_i^2 + 2i\alpha_r\alpha_i\right) + (\alpha_r + \alpha_i)\sqrt{2\omega}q' - \tfrac{1}{2}\omega q'^2\right]$$

$$= \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} \exp\left[-2\alpha_r^2 + 2\alpha_r\sqrt{2\omega}q' - \omega q'^2 + \text{imaginary part}\right]$$

$$= \left(\frac{\omega}{\pi}\right)^{\frac{1}{4}} \exp\left[-\omega(q'^2 - 2\sqrt{\frac{2}{\omega}}\alpha_r q' + \frac{2}{\omega}\alpha_r^2\right]$$

and quadrature distribution is $\left|\langle q'|\alpha\rangle\right|^2 = \left(\frac{\omega}{\pi}\right)^{\frac{1}{2}} \exp\left[-\omega(q' - \sqrt{\frac{2}{\omega}}\alpha_r)^2\right]$

This is Gaussian wave packet which is centered at $q' = \sqrt{\frac{2}{\omega}}\alpha_r$. Coherent state can be expressed in terms of Gaussian wave packet.

Hermitian Operator X_1 and X_2:

For the physical meaning of the canonical momentum operator p and canonical operator q of the field, we introduce two new hermitian operators X_1 and X_2 as

$X_1 = \sqrt{\frac{2\omega}{\hbar}}\, q$, $X_2 = \sqrt{\frac{1}{2\omega\hbar}}\, p$, in quantum unit $\hbar = 1$ then $X_1 = \sqrt{2\omega}\, q$, $X_2 = \sqrt{\frac{1}{2\omega}}\, p$

$$a = X_1 + i X_2, \; a^+ = X_1 - i X_2$$

$$X_1 = \frac{1}{2}(a + a^+), \quad X_2 = \frac{1}{2i}(a - a^+) \qquad (1)$$

Which gives $[X_1, X_2] = \frac{i}{2}$. For single mode radiation vector potential is

$$\vec{A} = \frac{1}{\sqrt{2\omega V}}\varepsilon\left[a e^{i(\vec{k}.\vec{r}-\omega t)} + a^+ e^{-i(\vec{k}.\vec{r}-\omega t)}\right]$$

$$= \frac{1}{\sqrt{2\omega V}}\varepsilon\left[(X_1 + i X_2)e^{i(\vec{k}.\vec{r}-\omega t)} + (X_1 - i X_2)e^{-i(\vec{k}.\vec{r}-\omega t)}\right]$$

$$= \frac{1}{\sqrt{2\omega V}}\varepsilon\left[X_1\left(e^{i(\vec{k}.\vec{r}-\omega t)} + e^{-i(\vec{k}.\vec{r}-\omega t)}\right) + iX_2\left(e^{i(\vec{k}.\vec{r}-\omega t)} - e^{-i(\vec{k}.\vec{r}-\omega t)}\right)\right] \qquad (2)$$

$$= \sqrt{\frac{2}{\omega V}}\,\varepsilon\left[X_1\cos(\vec{k}.\vec{r} - \omega t) + X_2\sin(\vec{k}.\vec{r} - \omega t)\right]$$

Which shows that X_1 and X_2 are the amplitude operators of the filed whose phase are orthogonal. Thus X_1 and X_2 are called quadrature operators.

Uncertainties in X_1 and X_2 for coherent state:

Let ΔX_1 and ΔX_2 be the uncertainty in X_1 and X_2 respectively. ΔX_1 and ΔX_2 are square root of the variances of X_1 and X_2 are defined as follows:

$$\Delta X_1 = \sqrt{\left\langle X_1^2 \right\rangle - \left\langle X_1 \right\rangle^2}$$

$$\Delta X_2 = \sqrt{\left\langle X_2^2 \right\rangle - \left\langle X_2 \right\rangle^2}$$

For

$$\begin{aligned}
\langle X_1 \rangle &= \langle \alpha | X_1 | \alpha \rangle \\
&= \langle \alpha | \frac{1}{2}(a^\dagger + a) | \alpha \rangle \\
&= \frac{1}{2}\left\{ \langle \alpha | a^\dagger | \alpha \rangle + \langle \alpha | a | \alpha \rangle \right\} \\
&= \frac{1}{2}\left\{ \alpha^* + \alpha \right\}
\end{aligned}$$

Since $a|\alpha\rangle = \alpha|\alpha\rangle$ and $\langle \alpha | a^\dagger = \langle \alpha | \alpha^*$

Since $\alpha = \alpha_r + i\alpha_i$ then $\alpha^* + \alpha = 2\alpha_r$ hence

$$\langle X_1 \rangle = \alpha_r$$

$$\begin{aligned}
\left\langle X_1^2 \right\rangle &= \langle \alpha | X_1^2 | \alpha \rangle \\
&= \langle \alpha | \frac{1}{4}(a^\dagger + a)^2 | \alpha \rangle \\
&= \frac{1}{4}\left\{ \langle \alpha | a^{\dagger 2} | \alpha \rangle + \langle \alpha | a^2 | \alpha \rangle + \langle \alpha | (aa^\dagger + a^\dagger a) | \alpha \rangle \right\} \\
&= \frac{1}{4}\left\{ \langle \alpha | a^{\dagger 2} | \alpha \rangle + \langle \alpha | a^2 | \alpha \rangle + \langle \alpha | (2a^\dagger a + 1) | \alpha \rangle \right\} \\
&= \frac{1}{4}\left\{ \alpha^{*2} + \alpha^2 + 2\alpha^* \alpha + 1 \right\} \\
&= \frac{1}{4}\left\{ (\alpha^* + \alpha)^2 + 1 \right\} \\
&= \frac{1}{4}\left\{ 4\alpha_r^2 + 1 \right\} = \alpha_r^2 + \frac{1}{4}
\end{aligned}$$

hence variance of X_1

$$(\Delta X_1)^2 = \left\langle {X_1}^2 \right\rangle - \langle X_1 \rangle^2 = \frac{1}{4}$$

Similarly we can calculate

$$\left\langle X_2 \right\rangle = \alpha_i \ , \ \left\langle {X_2}^2 \right\rangle = \alpha_i^2 + \frac{1}{4} \text{ .and}$$

variance of X_2 ; $(\Delta X_2)^2 = \left\langle {X_2}^2 \right\rangle - \langle X_2 \rangle^2 = \frac{1}{4}$.Thus,

$$(\Delta X_1)^2 (\Delta X_2)^2 = \frac{1}{16} \text{ So, } (\Delta X_1)(\Delta X_2) = \frac{1}{4}.$$

This is the minimum value allowed by Heisenberg uncertainty principle. Hence, Coherent state is a minimum uncertainty state.